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Generalized Voter Models[1]

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We examine a one-dimensional class of interacting particle systems which generalize some voter models. This class includes a particular case in the class of models of catalytic surfaces introduced by Swindle and Grannan. We show that this class has the "clustering" property of ordinary finite-range voter models, at least when one is concerned with translation-invariant measures on the state space.

KEY WORDS: Interacting particle systems; martingales; voter models; catalytic surfaces.

INTRODUCTION

In this note we consider a class of one-dimensional particle systems that share properties of the finite-range voter model. We prove that this class cannot have nontrivial stationary measures which are translation invariant. Of course more is known about the voter models themselves, but in our more general context we are unable to find a nice duality. The class of particle systems includes the critical one-dimensional catalytic surface considered by Grannan and Swindle.[(4)] It was this paper that motivated this note.

The voter model with finite range has flip rates

$$c(x, \eta) = \sum_{y} p(x, y) I_{\{\eta(x) \neq \eta(y)\}}$$

where $p(\cdot, \cdot)$ represents the transition probabilities of a random walk with bounded jumps. Our generalizing processes are defined by the following properties:

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1. "Discrete, bounded" state space: The state space will be of the form D^Z for D some finite subset of R.

2. Finite range: Let $c(x, k, \eta)$ denote the rate at which η flips to $\eta(x, k)$, where

$$\eta(x, k)(y) = \eta(y) \qquad \text{if} \quad y \neq x; \qquad \eta(x, k)(x) = k$$

Then

$$c(x, k, \eta) = f(\eta(x-K), \eta(x-K+1), \ldots, \eta(x+K))$$

for some K and function from $D^{2K+1} \to R$, both not depending on x.

3. Martingale: Denote the expected size of a flip at site x by

$$e(x, \eta) = \sum_{k \in D} c(x, k, \eta)[k - \eta(x)]$$

There exists an absolute constant for the system, M, so that for any n, η

$$\left| \sum_{|x| \leqslant n} e(x, \eta) \right| \leqslant M$$

Equivalently,

$$\|\Omega f_n\|_\infty < M \qquad \forall n$$

where $f_n(\eta) = \sum_{i=-n}^{n} \eta(i)$.

4. Active: Let $D = \{r_1 < r_2 < \cdots < r_n\}$. If $\eta(x) = r_j$ with $1 < j < n$, then

$$\sum_{|y-x| \leqslant K} \sum_{k \in D} c(y, k, \eta) > 0$$

Also if $\{\eta(x), \eta(x+1)\}$ or $\{\eta(x), \eta(x-1)\} = \{r_1, r_n\}$, then

$$\sum_{|y-x| \leqslant K} \sum_{k \in D} c(y, k, \eta) > 0$$

Here K is as in property 2.

Property 3 is the property that we use most. It ensures that if $\{\eta_t : t \geqslant 0\}$ is a generalized voter model, then $\sum_{|x| \leqslant n} \eta_t(x)$ is a martingale plus some edge effects that become negligible as n becomes large.

We will prove the following result.

Theorem. For a generalized voter model the only stationary measures which are translation invariant are those which consist of point masses at the configurations of all r_1 or all r_n.

Remark. In this paper we will consider the point masses at all r_1 or all r_n and their convex combinations to be trivial measures. Condition 4 ensures that these are the only traps.

An immediate consequence of the Theorem follows:

Corollary. Consider a GVM with translation-invariant starting measure v. The Cesaro means $(1/T)\int_0^T P_s v\,ds$ converge in distribution to $\alpha\delta_{r_1} + (1-\alpha)\,\delta_{r_n}$, where $\alpha r_1 + (1-\alpha) r_n = E^v[\eta(0)]$.

Grannan and Swindle[4] consider a class of processes on X, the subset of $\{-1, 0, 1\}^{Z^d}$ consisting of configurations with no neighboring sites having opposite values. In one dimension these processes have flip rates given as follows.

If $\eta(x) = 1$:

$$c(x, -1, \eta) = 0$$

$$c(x, 0, \eta) = (1-p)(I_{\eta(x-1)=0}(1 + I_{\eta(x-2)\neq 1}) + I_{\eta(x+1)=0}(1 + I_{\eta(x+2)\neq 1}))$$

If $\eta(x) = -1$:

$$c(x, 1, \eta) = 0$$

$$c(x, 0, \eta) = p(I_{\eta(x-1)=0}(1 + I_{\eta(x-2)\neq -1}) + I_{\eta(x+1)=0}(1 + I_{\eta(x+2)\neq -1}))$$

If $\eta(x) = 0$:

$$c(x, 1, \eta) = pI_{\eta(x-1),\,\eta(x+1)\neq -1}, \qquad c(x, -1, \eta) = (1-p)I_{\eta(x-1),\,\eta(x+1)\neq 1}$$

It was shown that if the parameter p is not equal to $\frac{1}{2}$, then the only stationary measures are the trivial measures. Our result shows that in the case $p = \frac{1}{2}$, the only translation-invariant stationary measures are trivial.

In Section 1 we show that $f_n(\eta_t)$ can be written as a martingale plus a term which changes with a bounded rate, not depending on n. Second we show that if a nontrivial invariant measure μ exists, then we may assume without loss of generality that there is a $c \in (r_1, r_N)$ so that

$$\mu\text{-a.s.} \quad \lim_{n\to\infty} \frac{1}{2n} f_n(\eta) = c \qquad (*)$$

In Section 2 we complete our proof with an argument by contradiction. We show that for any translation-invariant, stationary measure satisfying (*), there is "too much life" in the martingale part of f_n for (*) to hold at all subsequent times.

1. We assemble some preliminary observations. Since our state space is D^Z, where $D=\{r_1<r_2<\cdots<r_n\}$, every possible jump in the value of $\eta_t(x)$ as t varies lies in the finite set $J=\{r_i-r_j,\ 1\leqslant i,j\leqslant n,\ i\neq j\}$. We define $j_{\min}$ to be $\min\{|j|: j\in J\}$ and $j_{\max}$ to be $\max\{|j|: j\in J\}$ ($\leqslant r_n-r_1$).

Recall that $\sum_{i=-n}^{n}\eta(i)=f_n(\eta)$.

Lemma 1.1. The jump process

$$f_N(\eta_t)=\sum_{j=-N}^{N}\eta_t(j)$$

can be decomposed as three jump processes:

$$f_N(\eta_s)-f_N(\eta_0)=Y_s^N+Z_s^N+W_s^N$$

where all processes on the right-hand side are initially zero and (1) Y_s^N is a martingale with respect to the natural filtration of the GVM η_s. It does not share any jumps with the processes Z^N and W^N; (2) Z_s^N is equal to $j_{\max}$ times a Poisson process running at rate $M/j_{\min}$; and (3) $-W_s^N$ is increasing and stochastically less than $j_{\max}$ times a Poisson process of intensity $2M/j_{\min}$.

Proof. Let $J=\{j_1, j_2,\dots, j_p\}$. At time s the rate at which $f_N(\eta_s)$ makes a jump of size j_r equals $d(\eta_s, r)$ for some measurable set of functions $d(\cdot, r)$. Property 3 of GVMs requires that

$$\left|\sum_{r=1}^{p} d(\eta_s, r)\, j_r\right|<M$$

From this we see easily that the p-vector $(d(\eta_s, 1), d(\eta_s, 2),\dots, d(\eta_s, p))$ can be written as $R_s^Y+R_s^O$, where

(a) $\sum_{r=1}^{p}(R^Y)_r\, j_p=0$

(b) $|\sum_{r=1}^{p} R_r^O|=\sum_{r=1}^{p}|R_r^O|\leqslant M$

We now take Y_s^N to be the jump process corresponding to jump rate R_s^Y. It is obvious that $f_N(\eta_s)-f_N(\eta_0)-Y_s^N$ can be decomposed into $W_s^N+Z_s^N$. ∎

Remark. We can and will suppose that the jump rates of Y_s^N are measurable functions of η_s.

Let S^x denote the set of translation-invariant probability measures on D^Z such that

$$\mu\text{-a.s.}\quad \lim_{n\to\infty} \frac{1}{2n} \sum_{i=-n}^{n} \eta(i) = x$$

The proof given below is due to Tom Liggett and replaces the longer and clumsier original.

Proposition 1.2. If there exists a nontrivial stationary, translation-invariant probability measure μ, then for some $x \in (r_1, r_n)$ there exists a stationary v in S_x.

Proof. Let μ be a nontrivial stationary probability measure which is translation invariant. The translation invariance ensures (see, e.g., ref. 7) that

$$\mu\text{-a.s.}\quad \lim_{n\to\infty} \frac{1}{2n} f_n(\eta) = G(\eta) \text{ exists}$$

and that

$$\mu = \int_{r_1}^{r_n} d\chi(x)\, v^x \tag{$*$}$$

where χ is some probability measure and $v^x \in S_x$ for each x. The measures v^x can be thought of as conditional distributions of η given $G(\eta)$. Since μ is nontrivial, the probability measure χ must put positive mass on (r_1, r_n).

From the basic properties of operators of Markov processes (see, e.g., ref. 6, p. 16),

$$P_t f_N(\eta) - f_N(\eta) = \int_0^t P_s(\Omega(f_N(\eta)))\, ds$$

Property 3 defining generalized voter models ensures that $\|\Omega(f_N)\|_\infty \leqslant M$ and so $\|P_t f_N - f_N\|_\infty \leqslant Mt$. Therefore

$$\begin{aligned} E^\mu&\left[\left(\frac{1}{2N} f_N(\eta_t) - \frac{1}{2N} f_N(\eta)\right)^2\right] \\ &= E^\mu\left[\left(\frac{1}{2N} f_N(\eta_t)\right)^2\right] - E^\mu\left[\left(\frac{1}{2N} f_N(\eta)\right)^2\right] \\ &\quad + 2E^\mu\left[\frac{1}{2N} f_N(\eta)\left(\frac{f_N(\eta)}{2N} - P_t\frac{f_N(\eta)}{2N}\right)\right] \leqslant \frac{Kt}{N} \end{aligned}$$

for some K not depending on N. (Since μ is stationary, the first two terms on the right-hand side vanish.) Letting N tend to infinity, we deduce that μ-a.s. $G(\eta)=G(\eta_t)$ for all t. From this fact it is standard to deduce that for χ-a.a. x, $P_t v^x = v^x$. Therefore, by Fubini's theorem, for χ-a.a. x, $P_t v^x = v^x$ for Lebesgue almost all positive t. Our processes have bounded flip rates, so this entails that for χ-a.a. x, $P_t v^x = v^x$ for all t.

As χ is not concentrated on the points $\{r_1, r_n\}$, this proves the proposition. ∎

2. In this section we argue that there can be no nontrivial invariant measures for a GVM process. As mentioned in the Introduction, we assume the converse. We may assume that there exists a measure as in Proposition 1.2. We argue that the martingale square bracket process $\langle Y_t^N \rangle$ must be of order tN and that therefore, by the martingale central limit theorem, there will be a nontrivial chance that Y_{sN}^N will be of order $\sqrt{s}\,N$. If s is chosen correctly, then, we show, this discrepancy cannot be balanced by $W_{sN}^N + Z_{sN}^N$ and so $(1/2N)\, f_N(\eta_{sN})$ will differ from $(1/2N)\, f_N(\eta_0)$ significantly. This is the desired contradiction.

We begin this section by assuming the falsity of the Theorem, which is to say that we assume the existence of a nontrivial translation-invariant, stationary measure. As has already been stated, Proposition 1.2 enables us to argue that if there were such a measure, then there would have to be an invariant measure μ in S^x for some $x \in (r_1, r_n)$. Let $c(r)$ equal

$$\sum_{r_i - r_k = j_r} \int c(0, r_i, \eta) I_{\{\eta(0) = r_k\}} v(d\eta)$$

By Birkhoff's ergodic theorem,

$$\lim_{n \to \infty} \frac{1}{2N} \sum_{x=-N}^{N} \sum_{h \in D} c(x, h, \eta)[h - \eta(x)]^2 = \sum_{r=1}^{p} c(r)(j_r)^2 = k(\eta)$$

for v almost all η. The defining properties of GVMs force this limit to be strictly positive. It should be noted that we cannot suppose that this limit is necessarily constant, however, since the extreme points of the set of stationary, translation-invariant measures need not be ergodic. See ref. 1 for a nice counterexample. Therefore, for given $\delta > 0$, we can find $k > 0$ so that for N large enough and all t,

$$P^v \left[\int_0^t \frac{1}{2N} \sum_{x=-N}^{N} \sum_{h \in D} c(x, h, \eta_s)(h - \eta_s(x))^2 \, ds < kt \right] < \delta \tag{A}$$

We will use the decomposition of Lemma 1.1:

$$f_N(\eta_s) - f_N(\eta_0) = Y_s^N + Z_s^N + W_s^N$$

The lemma below follows simply from Theorem 3.2 of Hall and Heyde (ref. 5, p. 52).

Lemma 2.1. For any v, define the stopping time

$$S(v) = \inf\{s: \langle Y^N \rangle_s > v\}$$

where $\langle Y^N \rangle_s$ is the square bracket process for the martingale Y_s^N. Then

$$\frac{Y_{S(v)}^N}{\sqrt{v}} \xrightarrow{D} N(0, 1)$$

as v tends to infinity.

Consider the compensator of the process $\langle Y_s^N \rangle$. This is equal to a process $\int_{u=0}^{s} W_u^N \, du$, where

$$\left| W_u^N - \sum_{x=-N}^{N} \sum_{v \in D} c(x, v, \eta_u)[v - \eta_u(x)]^2 \right| < Mj_{\max}^2$$

Consequently, given inequality (A), for given $\delta > 0$, we can find $k > 0$ so that for N large enough

$$E^v \left[\int_0^t \frac{1}{2N} W_u^N \, du < kt \right] < \delta \qquad \forall t$$

From this and from the strong law of the large numbers property of the simple Poisson process it follows that for t positive and bounded away from 0,

$$\frac{|\langle Y_t^N \rangle - \int_{u=0}^{t} W_u^N \, du|}{\int_{u=0}^{t} W_u^N \, du}$$

tends to zero as N tends to infinity. Thus, there is a k_0 so that for any $t \geq 1$,

$$P^v[\langle Y_t^N \rangle \langle k_0 tN] < \tfrac{1}{5} \tag{B}$$

We now choose and fix γ and $\varepsilon \ll 1$ so that

(1) $\int_{-\infty}^{-\gamma} 1/(2\pi)^{1/2} \, e^{-x^2/2} \, dx > \frac{2}{5}$

(2) $\varepsilon((M+1)(r_n - r_1)/j_{\min}) < (\gamma/16)(k_0\varepsilon)^{1/2} < (r_n - r_1)/16$

Lemma 2.2. Define the stopping time $T = \inf\{t: Y_t^N < -N\gamma(k_0\varepsilon)^{1/2}\}$. The probability that T is less than εN exceeds $\frac{1}{5}$ for all N large enough.

Proof. Lemma 2.1 ensures that as N tends to infinity,

$$\frac{Y_{S(k_0\varepsilon N^2)}}{(k_0\varepsilon N^2)^{1/2}}$$

tends to a standard normal in distribution. Our choice of γ and inequality B ensure that for N large enough

$$P[Y_{S(k_0\varepsilon N^2)} < -N\gamma(k_0\varepsilon)^{1/2}] > \tfrac{2}{5}$$

which certainly implies that $P[T \leqslant S(k_0\varepsilon N^2)] > \frac{2}{5}$. But our choice of k_0 implies that $P[S(k_0\varepsilon N^2) \leqslant N\varepsilon] > \frac{4}{5}$. The lemma follows. ▌

The lemma below follows from general theory (see, e.g., ref. 3).

Lemma 2.3. Let V_t be a continuous-time martingale whose jumps are bounded by δ. If $a < c < b$ and $V_0 < c$, then

$$P[\inf\{t\colon V_t \leqslant a\} < \inf\{t\colon V_t \geqslant b\}] \geqslant \frac{b-c}{b+\delta-a}$$

Proof of Theorem. We apply Lemma 2.3 to the martingale Y^N_{T+s} conditioned on the event $T \leqslant \varepsilon N$, taking a to equal $-4N(r_n - r_1)$, b to equal $-N\gamma 2(k_0\varepsilon)^{1/2}$, and c to equal $-N\gamma(k_0\varepsilon)^{1/2}$. We conclude that with probability greater than

$$\frac{1}{5}\,\frac{\frac{1}{2}\gamma(k_0\varepsilon)^{1/2}}{4(r_n - r_1) + j_{\max}/N - \frac{1}{2}\gamma(k_0\varepsilon)^{1/2}}$$

either $Y^N_{\varepsilon N} < -N(\gamma/2)(k_0\varepsilon)^{1/2}$ or, for some $s \leqslant N\varepsilon$, $Y^N_s < -4N(r_n - r_1)$. We will secure the desired contradiction by showing that unlike the probability in question, both these later events have a probability which tends to zero as N tends to infinity.

First consider the event $\{Y^N_{\varepsilon N} < -N(\gamma/2)(k_0\varepsilon)^{1/2}\}$. By the decomposition of Lemma 1.1, however, on this event either

$$\text{(i)}\quad \frac{f_N(\eta_{N\varepsilon}) - f_N(\eta_0)}{N} \leqslant \frac{Y^{N_{N\varepsilon}}}{N} + \frac{Z^{N_{N\varepsilon}}}{N} \leqslant -\varepsilon$$

or

$$\text{(ii)}\quad Z^N_{N\varepsilon} \geqslant 2N\varepsilon\,\frac{Mj_{\max}}{j_{\min}}$$

The first event has probability tending to zero as N tends to infinity, since with respect to P_ν, $f_N(\eta_0)/N$ [and therefore $f_N(\eta_{N\varepsilon})/N$, since ν is invariant] tends to $2x$ in probability as N tends to infinity. That the probability of the second event goes to zero is a simple consequence of the law of large numbers.

Now consider the event $\{\text{for some } s \leqslant N\varepsilon,\ Y_s^N < -4N(r_n - r_1)\}$. Since $f_N(\eta_s) - f_N(\eta_0) \geqslant -(2N+1)(r_n - r_1)$, and W_s^N is negative, the event in question can only occur if $Z_{\varepsilon N}^N \geqslant (2N-1)(r_n - r_1) \geqslant N\varepsilon(2M+1)\, j_{\max}/j_{\min}$ for N large enough. Again the law of large numbers ensures that this has probability tending to zero as N tends to infinity.

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